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a Bethe lattice

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P D Gujrati

Department of Physics, Department of Polymer Science, University of Akron, Akron, OH 44325, USA

Abstract
We consider the problem of correlated percolation on a Husimi cactus, which allows finite loops of size l, to investigate the effects of loop formation on percolation properties. In particular, we calculate how the percolation threshold and the percolation probability depend on l and the loop activity n. We calculate the contribution and its dependence on l and n from finite and infinite clusters to all densities. We show that macroscopic loops are formed immediately after percolation, and we calculate their density dependence on l and n. We compare the results on Husimi cactus with those on a Bethe lattice. We finally establish that the Husimi cactus turns into a Bethe lattice as $l \rightarrow \infty$.

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1. Introduction

The tree approximation in *random* percolation and branched polymers has been extensively used for almost six decades [1–4], and has played a major role in theoretical physics [5] as an important aspect of disorder because of the analytical solution in this approximation. The clusters of particles (lattice animals), bonded by chemical interactions or nearest-neighbour physical contacts, can be thought of as representing branched polymers. Indeed, the field of random percolation has evolved into a well-established and rigorous endeavour [6–10]. The tree approximation treats branched polymers or clusters as *trees*; no *loops* or *cycles* are allowed. The activity for each of the allowed functionalities must be *exactly* equal to one, and there must be no interaction of any sort, except for excluded volume effects, and polymer bonding. However, the predictions of the two versions of the tree approximation due to Flory [1], and Stockmayer [2] in the postgel regime are *very* different and somewhat contradictory. Flory argued for the presence of loops in the postgel regime, while Stockmayer argued for their absence. The situation was clarified only recently [11], when it was shown that both were partially correct. Stockmayer was correct in that there are no finite size loops,
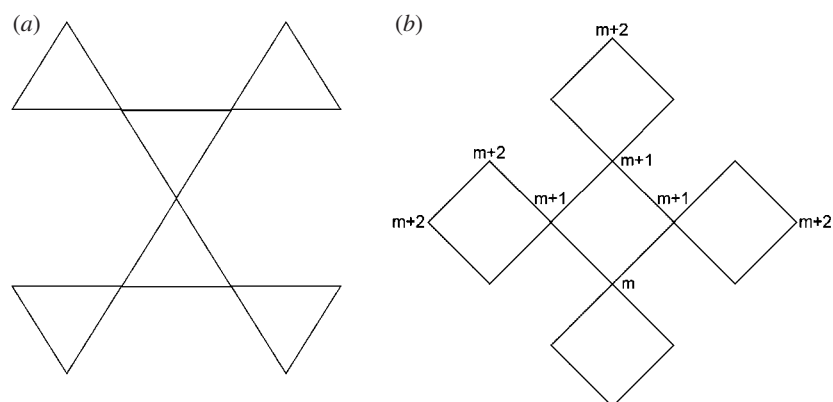
Figure 1. (*a*) A triangular cactus made up of triangles, and (*b*) a square cactus made up of squares, connected in a tree-like manner, so there is only one connected path between any two polygons. The corners of the polygons are labelled by a generation number m.

while Flory was correct in that there are macroscopic loops, in the postgel regime in the tree approximation. It was pointed out there that the conventional tree approximation [1, 2] on a regular lattice can be *different* from random percolation on a Bethe lattice of the same coordination number. In particular, the loop activity n, which must be zero for the tree approximation (thus defining the tree percolation) and equal to one for random percolation (thus defining the random percolation), plays no role in the Bethe lattice calculation as there are *no* finite size loops in the Bethe lattice. This makes the tree and random percolation the same on a Bethe lattice. Thus, this calculation does not really settle how the tree and the random percolation are different. In addition, in polymerization of branched polymers, one usually deals with only lower functionalities like tri-, and tetra-functionalities [3, 4], even though the lattice required to represent the system will usually have a coordination number q somewhat larger than 4. Random percolation either on a Bethe lattice or a regular lattice requires *all* functionalities up to the coordination number q of the lattice; see equation (3*c*) below. If the maximum allowed functionality is strictly less than q, the percolation is no longer random in nature, but becomes *correlated* or *non-random*. We refer the reader to [11, 12] for further discussion on these points, and for an extensive bibliography.

On a regular lattice, which allows for loops, the tree percolation should be very different from random percolation. This is so because the requirement that no loops are ever formed generates *correlations* in the placement of chemical or physical bonds on the lattice. There are no such correlations in random percolation. Thus, the tree percolation should have different predictions than random percolation on lattices having loops. It is the differences that we wish to investigate here. Since the investigation cannot be carried out analytically for regular lattices at the present moment, we must resort to an alternative approach.

In the present work, our aim is to investigate analytically the effects of the loop activity n on a Husimi cactus (see figures 1(*a*), (*b*) for a triangular and a square cactus), which allows for small-size loops. We will consider the case in which *two l-sided regular polygons meet at each site of the cactus including the origin. (In general, more than two different polygons meet at each site and our approach is easily extended to the general case.) Each polygon has l other polygons connected to it, one at each of its l corners, in such a way that there are no closed loops of size larger than l, the loop formed by each polygon. We consider two different Husimi cacti, made up of triangular and square polygons, respectively. But the method is

easily generalized to higher polygon cacti, and we give results for a general Husimi cactus. We consider not only the random percolation ($n = 1$), and the tree percolation ($n = 0$), but the more general problem with arbitrary n on the cactus. The calculation is done exactly using the recursive technique that has been used elsewhere [11–13] to study percolation on a Bethe lattice. The analytical results are compared with those on the Bethe lattice. We also discuss the relationship between a Husimi cactus and a Bethe lattice as $l \to \infty$.

The layout of the paper is as follows. We introduce our model and notation in the next section. The recursion relations (RRs) are derived in sections 3 and 4 for a triangular and a square cactus, respectively. However, we present final results in a form valid for a general cactus of l-sided polygons. The general cactus is treated in section 5. We also consider random and tree percolation here in detail. The percolation threshold, percolation probability and other percolation properties are introduced and investigated for a triangular cactus in section 6. We consider arbitrary l in section 7. The final section contains a discussion and a brief summary of our results.

2. Model

The general model of branched polymers in a solution is defined in [11, 12], and requires a set of branching activities w_k for a k-functional unit, $k = 1, 2, \ldots$, the bond activity K, the loop activity n, and the Boltzmann weight w for the nearest-neighbour interaction between a monomer and a solvent molecule. The notation is introduced in [11, 12], which we closely follow here. We will use 'bond' or 'polymer bond' to denote chemical or physical bonding; the latter is useful for (site) percolation in monomeric units. We will use the term 'lattice bond' to denote a bond on the lattice, which may or may not be 'occupied' by a polymer bond. We will use 'percolation' in this work to contain not only random percolation ($n = 1$) and the tree percolation ($n = 0$), but all other values of n. For the percolation considered in this work, we additionally set $w_k = 1$ for each k-functional unit, and $w = 1$. Therefore, for $n = 1$, percolation considered here reduces to the random percolation. The technique is easily extended to arbitrary branching activities, and will be discussed elsewhere. The total partition function (PF) for percolation is given by

$$Z_N(K|n) = \sum_{B,L} \Omega(B, L) K^B n^L \tag{1}$$

where B and L denote the total number of bonds and loops (or cycles) in all polymers or clusters on a lattice of total sites N; the quantity $\Omega(B, L)$ denotes the number of distinct configurations, each having the same B and L. In addition, each cluster must have *at least* one bond. Any site left uncovered by polymers is considered occupied by a solvent molecule. In this model, the monomer and the solvent molecule are supposed to occupy the same volume. Let P denote the number of distinct polymers, N_0 the number of solvent molecules, M the total number of monomers, i.e. sites covered by all polymers, and V_k the number of k-functional branches in any configuration. For each configuration, the following geometrical identities must be observed for all clusters on a *finite* lattice:

$$2B = \sum_{k \geqslant 1} k V_k \qquad M = N - N_0 \qquad \sum_{k \geqslant 1} (2 - k) V_k = 2(P - L). \tag{2a}$$

The first equation shows that each k-functional branch contributes k bond, and that each bond is counted twice in the sum. The second equation merely states that each site is covered by a monomer, or by a solvent molecule. The last equation, also known as the Euler relation for graphs, is taken as a definition of the number L of loops. The thermodynamic limit is obtained by taking $N \to \infty$ at fixed activities. In this limit, various extensive quantities also

diverge, but their ratio with N remains finite. Let us introduce these densities $\phi_\alpha = \alpha/N$ in the thermodynamic limit, where α represents any extensive quantity (B, P, L, M, N_0, V_k, etc). For convenience, we will use ϕ_0 and ϕ_k for $\alpha = N_0$ and V_k, respectively. In terms of densities, we can re-express equation (2a) as follows:

$$2\phi_B = \sum_{k \geqslant 1} k\phi_k \qquad \phi_M = 1 - \phi_0 \qquad \sum_{k \geqslant 1}(2 - k)\phi_k = 2(\phi_P - \phi_L). \quad (2b)$$

As discussed in [11, 12], the last equation in (2b) relates local densities ϕ_k to global densities ϕ_P and ϕ_L, and remains valid as long as percolation has not occurred. After percolation, this equation need not be satisfied for all clusters due to the emergence of the infinite cluster. However, the equation remains valid when applied separately to finite and infinite clusters. We refer the reader to [11, 12] for further details. The bond occupation probability p is related to the bond density via

$$p = \phi_B/(q/2) \qquad\qquad\qquad\qquad\qquad\qquad (2c)$$

since the number of lattice bonds for a lattice of the coordination number q is $N_B = qN/2$.

For random percolation ($n = 1$), we can introduce

$$\Omega(B) = \sum_{L:B} \Omega(B, L) \qquad\qquad\qquad\qquad\qquad (3a)$$

where the sum is over all allowed loops consistent with B bonds in the configuration. Since bonds are now randomly placed on the lattice, we evidently have

$$\Omega(B) = \binom{N_B}{B}. \qquad\qquad\qquad\qquad\qquad (3b)$$

The partition function for random percolation[1], is

$$Z_N(K|1) \equiv \sum_B \Omega(B)K^B = (1 + K)^{N_B}. \qquad\qquad (3c)$$

The situation for tree or other non-random percolation is not so trivial, as $n \neq 1$ generates correlations in the placement of bonds on the lattice; we need to inquire if loops are formed, the determination of which cannot be done locally. Such correlations are absent for random percolation. A similar complication arises if all functionalities up to and including q are not allowed with activities equal to one.

The main interest in percolation is to determine the percolation threshold $K_c(n)$ and the percolation probability $P_n(p)$. The simple form in equation (3c) gives no hint of these quantities, which must be calculated by other methods. Here, we will employ the recursive method proposed in [11, 12], which requires slight modification. We will also evaluate the free energy by using the scheme proposed in [13].

The calculation is carried out for a Husimi cactus. The sites in the cactus are labelled by a generation index m, which increases gradually as we move outwards away from the origin ($m = 0$), as shown in figure 1(b). At each site at the mth generation, there are $q = 4$ nearest-neighbours, as shown in figure 2, equally shared by the two polygons. Of the two polygons meeting at the mth generation ($m > 0$), the polygon that contains sites of higher generation ($m + 1$) (lower generation ($m - 1$)) will be called the upper (lower) polygon. The origin is exceptional in that both polygons are upper polygons. With respect to the upper polygon at the mth site, the mth site will be called its *base* site, and the ($m \pm 1$)th sites its upper (lower) sites.

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[1] There is an inconsequential error in the exponent in equation (6) of [11]. It should be N_B, as given in equation (3c) here, and in equation (2.9) in [12]. The error does not affect any of the conclusions in [11].
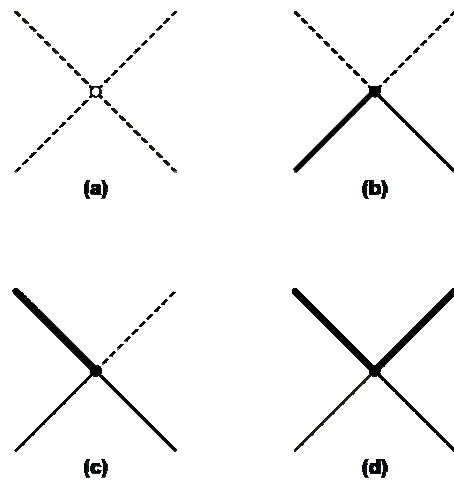
Figure 2. Various possible states σ of a site: (a) $\sigma = s$ representing the s-state with a solvent molecule at the site; (b) $\sigma = 0$ representing the 0-state, having no incident bonds from the upper polygon; (c) $\sigma = 1$ representing the 1-state with one incident bond from the upper polygon; (d) $\sigma = 2$ representing the 2-state with two incident bonds from the upper polygon. The broken bonds cannot be occupied by polymers. A thick bond represents an occupied bond. A thin bond may or may not be occupied.

The state σ of each vertex can be classified as s, 0, 1, 2 and b defined below.

(1) The state $\sigma = s$, see figure 2(a), means that the site is occupied by a solvent molecule (s). There cannot be any polymer bond attached to this site from either of the two polygons. This is indicated by broken bonds that cannot be occupied.

(2) The state $\sigma = 0$, see figure 2(b), means that there is no polymer bond incident on this site from the upper polygon. There must be at least one polymer bond (shown as thick) incident on this site from the lower polygon. The thinner bond indicates the possibility that it may or may not be occupied by a polymer bond.

(3) Figure 2(c) indicating one polymer bond and 2(d) indicating two polymer bonds incident on this site from the upper polygon represent $\sigma = 1$ and 2, respectively. The broken bond indicates that it cannot be occupied by a polymer bond. There may or may not be any bonds incident on this site from the lower polygon; both lower bonds are thin. The state $\sigma = b$ represents $\sigma = 1$ and 2 collectively.

The tree-like structure of the cactus allows us to treat its various branches, which are obtained by cutting out some polygon, as independent. Thus, the method proposed in [13] enables us to calculate the total partition function and the free energy per site. For this, we consider cutting the cactus into two branches at some mth level site, and introduce the partial partition function $Z_m(\sigma)$, which represents the contribution of the entire branch containing the upper polygon at the mth level site, given that the base site is in the σ state. This partial partition function can be related in the recursive scheme with $Z_{m+1}(\tau)$ at the higher level as follows:

$$Z_m(\sigma) = \mathrm{Tr}_t \left[W(\sigma|t) \prod_{\tau \in t} Z_{m+1}(\tau) \right] \tag{4}$$
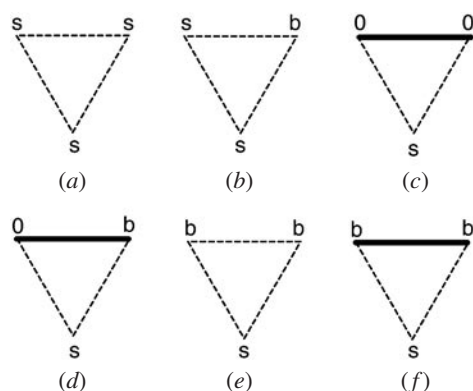
Figure 3. Possible states of a triangle with the base site in the s-state.

where t denotes the set of states $\{\tau\}$ of the $l' = l - 1$ upper sites in the polygon, and W represents the contribution to the partial partition function from the upper mth polygon, given that the base site is in the state σ and the upper sites are in the state set $\{\tau\}$. The trace operation denotes summing over all possible states of the $(l - 1)$ upper sites. In the following two sections, we will apply equation (4) to two different Husimi cacti. The results for arbitrary l will be presented in section 5.

The following combinations will prove extremely useful in the following:

$$U_m = Z_m(0) + Z_m(b) \qquad V_m = Z_m(s) + Z_m(b). \tag{5}$$

3. Recursive solution: triangular cactus ($l = 3$)

We consider the cactus in figure 1(a) composed of triangles. Let us first consider $\sigma = s$ at the mth site. There are two *upper* sites (at the $(m + 1)$th level), and we must consider the set t of the states of both sites. The sites can each be in the $\tau = s$ state (see figure 3(a)), which will contribute two factors of $Z_{m+1}(s)$. All three bonds in this triangle must be unoccupied, whenever there are at least two s sites in the triangle. Thus, there in no polymer bond and no loop formation, and the corresponding $W = 1$. The possibility $(s, 0)$ for the two upper sites is impossible, as all three bonds must be unoccupied in this polygon as there are two s sites; thus the state 0 with a bond in this polygon (which plays the role of the lower polygon at this site) is impossible. It is possible for the two sites to be in the (s, b) state (figure 3(b)). Here, $W = 2$ for the two possible ways a site can have the solvent molecule. Thus, the contribution is $2Z_{m+1}(s)Z_{m+1}(b)$. The two sites each can be in the 0-state (figure 3(c)), with the lattice bond between the two occupied by a polymer bond and contributing K to W. The contribution of this configuration is $KZ_{m+1}^2(0)$. Similarly, for the two sites in the $(0, b)$ state (figure 3(d)), the contribution is $2KZ_{m+1}(0)Z_{m+1}(b)$. The last possibility is (b, b). The bond between the two sites can be unoccupied (figure 3(e)) by a polymer bond and gives a contribution $W = 1$, or occupied by a polymer bond (figure 3(f)) and gives a contribution $W = K$. The two contributions can be added together to yield $W = u \equiv 1 + K$, so that the contribution becomes $uZ_{m+1}^2(b)$. We find that identical contributions are also obtained for $Z_m(0)$. Thus, we have

$$Z_m(s) \equiv Z_m(0) = Z_{m+1}^2(s) + 2Z_{m+1}(s)Z_{m+1}(b) + KZ_{m+1}^2(0)$$
$$+ 2KZ_{m+1}(0)Z_{m+1}(b) + uZ_{m+1}^2(b). \tag{6a}$$

We conclude that $U_m = V_m$ in the current model. Hence, we will use U_m for both in the following unless necessary. Using U_m, we can rewrite equation (6a) in a compact form ($l = 3$), which we will see later to be valid for any l:

$$Z_m(s) \equiv Z_m(0) = u^{l-2}U_{m+1}^{l-1}. \tag{6b}$$

We now consider the mth site state $\sigma = b$ and its two parts $\sigma = 1$ and $\sigma = 2$. The two upper sites cannot both be in the s-state since this requires all three bonds to be unoccupied; see above. The following combinations are allowed to occur: $(s, 0)$, (s, b), $(0, 0)$, $(0, b)$ and (b, b) in various possible orderings, as shown in figure 4. The calculation of corresponding W factors is also trivial here. We illustrate this with a few examples. Consider the first case, figure 4(a), which can occur in two possible orderings $\{s, 0\}$ and $\{0, s\}$. In each ordering, there must be a polymer bond attached to the 0-state site. The b-state at the mth level can have only one polymer bond due to the presence of an s-state site. The same is true of the case (s, b), figure 4(a). For both cases, $W = 2K$. For the case $(0, 0)$, we must consider three possibilities: there is one bond (figure 4(b)), or two bonds (figures 4(c), (d)) connected to the base site in the $\sigma = b$ state. There must be an additional bond between the two upper sites in figure 4(b); hence, it contributes $W = 2K^2$. The contributions of figures 4(c), (d) are K^2 and $K^3 n$, respectively, which can be combined together to contribute $K^2(1 + Kn)$. The presence of n signifies a loop formed by the three bonds, which contributes $K^3 n$. Thus, $W = K^2(3 + Kn)$. Similarly, we can easily check that W is $2K(1 + 3K + K^2 n)$ and $K(2 + 3K + K^2 n)$ for $(0, b)$ and (b, b), respectively; they can be obtained by replacing 0 by b in figures 4(c), (d). Finally, we have the recursion relation

$$
\begin{aligned}
Z_m(1) &= 2K[Z_{m+1}(s)Z_{m+1}(0) + Z_{m+1}(s)Z_{m+1}(b)] + 2K^2 Z_{m+1}^2(0) \\
&\quad + 4K(1 + K)Z_{m+1}(0)Z_{m+1}(b) + 2K(1 + K)Z_{m+1}^2(b) \\
&= 2Ku^{l-2}U_{m+1}^{l-1}
\end{aligned} \tag{7a}
$$

$$
\begin{aligned}
Z_m(2) &= K^2(1 + Kn)[Z_{m+1}^2(0) + 2Z_{m+1}(0)Z_{m+1}(b) + Z_{m+1}^2(b)] \\
&= K^2(u^{l-2} + vK^{l-2})U_{m+1}^{l-1}
\end{aligned}
$$

$$
\begin{aligned}
Z_m(b) &= 2K[Z_{m+1}(s)Z_{m+1}(0) + Z_{m+1}(s)Z_{m+1}(b)] + K^2(3 + Kn)Z_{m+1}^2(0) \\
&\quad + 2K(2 + 3K + K^2 n)Z_{m+1}(0)Z_{m+1}(b) + K(2 + 3K + K^2 n)Z_{m+1}^2(b) \\
&= KU_{m+1}^{l-1}[u^{l-2}(K + 2) + K^{l-1}v].
\end{aligned} \tag{7b}
$$

We have introduced $v = n - 1$. The final forms of various quantities in terms of l, given here and below, remain valid for any l, as we will see. We define the following ratios:

$$x_{m,s} \equiv Z_m(s)/Z_m(b) \qquad x_{m,k} \equiv Z_m(k)/Z_m(b) \qquad k = 0, 1, 2. \tag{8a}$$

From the definition, it is obvious that

$$x_{m,1} + x_{m,2} = 1. \tag{8b}$$

These ratios approach a fixed-point (FP) solution, which we denote without the index m, as we approach the origin of an infinite cactus. The condition in equation (8b) must remain valid at the FP. Moreover, in the case that we are considering, x_s and x_0 are the same and we will denote this common value by x in this work. Thus, we must calculate x and x_1; x_2 is given by $1 - x_1$; see equation (8b). Introducing a polynomial $Q_1(x, n)$ given by the ratio

$$Q_1(x, n) \equiv Z_m(b)/Z_{m+1}^{l-1}(b) \tag{9}$$

at the FP solution, we can write the FP equations after some trivial manipulation as

$$xQ_1 = u^{l-2}\xi^{l-1} \qquad x_1 Q_1 = 2Ku^{l-2}\xi^{l-1} \qquad x_2 Q_1 = K^2(u^{l-2} + K^{l-2}v)\xi^{l-1} \tag{10}$$
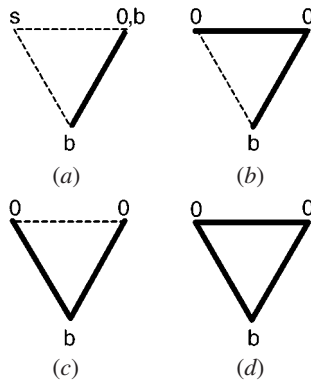
Figure 4. Possible states of a triangle with the base site in the b-state.

where we have introduced the useful combination

$$\xi \equiv x + 1. \tag{11a}$$

We have also made use of the following:

$$U_{m+1}/Z_{m+1}(b) \equiv V_{m+1}/Z_{m+1}(b) = \xi. \tag{11b}$$

The power of ξ in equation (10) is the number of upper sites $l - 1$ in the polygon.

The *physical significance* of the three equations in equation (10) is the following. Consider the mth site, which is occupied either by a solvent molecule or a monomer, but no polymer bonds are incident on it in the upper polygon. The remaining third bond connecting the higher generation sites may or may not be occupied and, thus, contributes a factor u. This explains the first equation. If the mth site is in the state $\sigma = 1$, then there are two ways this polymer bond in the upper polygon can be incident on the mth site. This contributes the factor $2K$. The third bond in the polygon contributes u as before. This explains the second equation. If the mth site is in the state $\sigma = 2$, then there are two polymer bonds in the upper polygon incident on the mth site. This contributes a factor K^2. The third bond in the polygon contributes $u + Kv \equiv (1 + Kn)$, where we have inserted the loop activity n for the loop that can be formed. This explains the last equation. We will find this discussion extremely useful when we generalize our results for arbitrary polygons in section 5. In terms of the combinations in equation (5), the above remarks translate into the following. Each upper site containing the solvent contributes a factor V_{m+1}; each upper site containing a monomer contributes a factor U_{m+1}.

By adding the last two equations in (10), we find that

$$Q_1(x, n) = K[(K + 2)u^{l-2} + K^{l-1}v]\xi^{l-1} \tag{12}$$

where we have used the fact that $x_1 + x_2 = 1$. Thus, the FP solution for $l = 3$ is analytically given by

$$x(n) = u/K[(K + 2)u + K^2v] \qquad x_1(n) = 2u/[(K + 2)u + K^2v]$$
$$x_2(n) = K(u + Kv)/[(K + 2)u + K^2v]. \tag{13}$$

For random percolation, we must set $n = 1$ or $v = 0$. We have

$$x = 1/K(K + 2) \qquad x_1 = 2/(K + 2) \qquad x_2 = K/(K + 2). \tag{14}$$

For tree percolation, we must set $n = 0$ or $v = -1$ and we have

$$x = (K + 1)/K(3K + 2) \qquad x_1 = 2(K + 1)/(3K + 2) \qquad x_2 = K/(3K + 2). \tag{15}$$

4. Square cactus ($l = 4$)

The calculation, though somewhat tedious, is almost identical to that for the triangular cactus. We label each site by a generation number, as shown in figure 1(b). Each site can be in any of the four possible states $\sigma = s, 0, 1$ and 2. The complication comes from having *three* higher level sites in a square, rather than the two upper sites in a triangle. The calculation requires considering all possible combinations of the states t of the three sites. These are: $t = (s, s, s)$, $(s, s, 0), (s, s, b), (s, 0, 0), (s, 0, b), (s, b, b), (0, 0, 0), (0, 0, b), (0, b, b)$ and (b, b, b) in all their possible orderings.

Let us consider the mth site in the s-state, and the three upper $(m + 1)$th level sites; see the central square in figure 1(b). In the following, we will consider the three upper sites in *clockwise* order when discussing the possible orderings of the states of these sites. The $(s, s, 0)$ is not possible in any of its permutations or orders. The possible orderings of $(s, 0, b)$ are $\{s, 0, b\}$, $\{s, b, 0\}$ and $\{0, s, b\}$, with outer states interchanged. The last ordering with its outer states 0 and b interchanged leaves 0 surrounded by two s state sites and is not possible. The remaining orderings with their outer states interchanged contribute $4K Z_{m+1}(s) Z_{m+1}(0) Z_{m+1}(b)$, with K accounting for a polymer bond attached to the 0-state site. Similarly, the ordering $\{0, s, 0\}$ is not possible for $(s, 0, 0)$. The contribution from $(s, 0, 0)$ is seen to be $2K Z_{m+1}(s) Z_{m+1}^2(0)$. Let us introduce a compact notation

$$Z_{m+1}(\tau, \tau', \tau'') \equiv Z_{m+1}(\tau) Z_{m+1}(\tau') Z_{m+1}(\tau'').$$

Here, τ, τ' and τ'' represent the states of the upper sites. We obtain the following RRs:

$$
\begin{aligned}
Z_m(s) \equiv Z_m(0) =\ & Z_{m+1}(s, s, s) + 2K Z_{m+1}(s, 0, 0) + 4K Z_{m+1}(s, 0, b) \\
& + (1 + 2u) Z_{m+1}(s, b, b) + K^2 Z_{m+1}(0, 0, 0) + K(K + 2u) Z_{m+1}(0, 0, b) \\
& + K(1 + 3u) Z_{m+1}(0, b, b) + u^2 Z_{m+1}(b, b, b) \\
Z_m(1) =\ & 2K[Z_{m+1}(s, s, 0) + Z_{m+1}(s, s, b) + K Z_{m+1}(s, 0, 0) + 2u Z_{m+1}(s, 0, b) \\
& + (1 + u) Z_{m+1}(s, b, b) + Ku Z_{m+1}(0, 0, 0) + K(K + 2u) Z_{m+1}(0, 0, b) \\
& + K(1 + 3u) Z_{m+1}(0, b, b) + u^2 Z_{m+1}(b, b, b)] \\
Z_m(2) =\ & K^2[Z_{m+1}(s, 0, 0) + 2 Z_{m+1}(s, 0, b) + Z_{m+1}(s, b, b) + (u^2 + K^2 v - 1) Z_{m+1}(0, 0, 0) \\
& + (3u^2 + 3K^2 v - 2) Z_{m+1}(0, 0, b) + (3u^2 + 3K^2 v - 1) Z_{m+1}(0, b, b) \\
& + (u^2 + K^2 v) Z_{m+1}(b, b, b)].
\end{aligned}
\tag{16}
$$

The RR for $Z_m(b)$ is obtained by adding the last two equations in equation (16). It is easy to check that these RRs can be rewritten in a compact form as follows: $Z_m(s) \equiv Z_m(0) = u^2 U_{m+1}^3$, $Z_m(1) = 2K u^2 U_{m+1}^3$, $Z_m(2) = K^2 U_{m+1}^3 (u^2 + K^2 v)$ and are equivalent to the general results in section 3 for $l = 4$. Similarly, if we introduce the ratios defined in equation (8a), and consider their FP solution, then they also satisfy the general relations in section 3 for $l = 4$.

It is easy to see that for $l = 4$, x and x_1 for random percolation are the same as in equation (14). However, they are different from those in equation (15) for tree percolation.

5. General solution for a l-polygon cactus

The general results for the FP ratios have already been given in equation (10), and can be derived by the general method for arbitrary l presented in section 7, where we investigate percolation related properties. From the first equation in equation (10), and the form of Q_1 in equation (12), we easily find that

$$\xi Q_1 = (u^l + K^l v) \xi^{l-1}. \tag{17}$$

The alternative form for Q_1 in equation (18) will prove more useful in the following:

$$Q_1 = (u^l + K^l v)\xi^{l-2}. \tag{18}$$

We now turn our attention to calculating various densities. First, we calculate the total partition function Z_N by considering the two polygons meeting at the origin:

$$Z_N \equiv Z_0^2(s) + 2Z_0(0)Z_0(b) + Z_0^2(b) = Z_0^2(b)\xi^2. \tag{19}$$

The density of solvent molecules is given by the ratio $Z_0^2(s)/Z_N$:

$$\phi_0 = (x/\xi)^2. \tag{20}$$

The density of monomers is $\phi_M = 1 - \phi_0$. The end-point density is obtained by considering all possible cases in which there is one end-point at the origin. This event corresponds to having the state of the origin in one polygon to be 0, and in the other to be 1. Thus, the ratio $2Z_0(0)Z_0(1)/Z_N$ gives the end-point density ϕ_1. We obtain

$$\phi_1 = 2xx_1/\xi^2. \tag{21}$$

Similarly, the ratios $[2Z_0(0)Z_0(2) + Z_0^2(1)]/Z_N$, $2Z_0(1)Z_0(2)/Z_N$ and $Z_0^2(2)/Z_N$ give ϕ_k, with $k = 2$, 3 and 4, respectively, and we obtain

$$\phi_2 = (2xx_2 + x_1^2)/\xi^2 \qquad \phi_3 = 2x_1x_2/\xi^2 \qquad \phi_4 = x_2^2/\xi^2. \tag{22}$$

It can be easily seen from equation (19) that the sum of all k-functional units ($k \geqslant 1$), which is nothing but the monomer density ϕ_M, and the solvent density ϕ_0 add up to one. This must be so as every site of the lattice is covered by either a solvent molecule or by a monomer. The bond density is obtained by the use of equation (2b). A simple algebra yields

$$\phi_M = 1 - x^2/\xi^2 \qquad \phi_B = (x_1 + 2x_2)/\xi. \tag{23}$$

Using equations (10) and (18), we can rewrite the two densities as a function of the two activities K and n only:

$$\phi_M = 1 - [u^{l-2}/(u^l + K^l v)]^2 \qquad \phi_B = 2K(u^{l-1} + K^{l-1}v)/(u^l + K^l v). \tag{24}$$

The loop density can also be easily calculated, as all finite loops are 'local' in that they are of the size of a polygon. It should be clear from the previous two sections that these loops contribute only when we consider the RR for $Z_m(b)$. Moreover, each loop comes with a factor of n. Thus, a loop is formed when all sides of a polygon are occupied, and contributes $K^l n U_{m+1}^{l'}$. The last factor $U_{m+1}^{l'}$ comes because each of the l' upper sites can be in either of the two states. The contribution to U_m from all configurations that do *not* contain any loop in the mth polygon is $U_m - K^l n U_{m+1}^{l'}$.

Consider the origin and the two polygons meeting there. If both polygons contain a loop, we obtain two loops and the contribution to the loop count is $2[K^l n U_1^{l'}]^2$. If only one of the polygons contains a loop, then the contribution is $2K^l n U_1^{l'}[U_0 - K^l n U_1^{l'}]$. We have multiplied this contribution by 2 to account for the two possible ways a polygon can be selected to have a loop. The sum of these two contributions divided by the total PF determines the loop density. Thus, we finally have

$$\phi_L \equiv 2n K^l U_1^{l'} U_0/l Z_N = 2n K^l \xi^{l-2}/l Q_1. \tag{25}$$

Here, we have used equations (9), and the fact that $U_m/Z_m(b) = \xi$ at the FP. We have also divided by the size of a polygon, since the same polygon contributes to the loop count for all of its l sites. Using equation (2b), we can calculate the number density ϕ_P. It should be remarked that the above loop density contains only finite size (l) loops. Using equation (18), we can rewrite the loop density as a function of the two activities as follows:

$$\phi_L = 2n K^l/[l(u^l + K^l v)]. \tag{26}$$

We now calculate the free energy for the model. For this, we use the trick proposed earlier [13]. For the trick to work, we also need to calculate the total PF of a smaller lattice. For the Husimi cactus, the smaller lattice is obtained by taking out the two central polygons meeting at the origin. This gives rise to $2l'$ branches from which l' smaller lattices can be constructed by joining two branches at a time. The partition function $Z_{N'}$ for the smaller lattice is given by exactly the same formula as in equation (19), except that all $Z_0(\sigma)$ are replaced by $Z_1(\sigma)$. Here, N' denotes the number of sites in the smaller lattice. There are $2l$ additional bonds or l additional sites in the larger lattice compared to those in all of the l' smaller lattices. Thus, the free energy *per site* (without the conventional $-T$, T being the absolute temperature in units of the Boltzmann constant) is given by

$$\omega \equiv (1/l)\ln(Z_N/Z_{N'}^2) = (2/l)\ln(Q_1/\xi^{l-2}) = (2/l)\ln(u^l + K^l \nu) \qquad (27)$$

where we have used equation (18) to derive the third equation. The last equation gives the free energy in terms of the two activities alone. It also shows that the free energy *depends* on the loop activity. In particular, it is smaller for the tree percolation than for the random percolation at the same K. Because of the presence of n in the PF in equation (1), we can also calculate the loop density by differentiating the free energy since $\phi_L = n(\partial\omega/\partial n)_K$. A simple manipulation shows that we obtain equation (26). We can also calculate the bond density in equation (24) by the relation $\phi_B = K(\partial\omega/\partial K)_n$.

The entropy $S(\phi_B, \phi_L)$ per site is obtained by the Legendre transform

$$S(\phi_B, \phi_L) = \omega(K, n) - \phi_B \ln K - \phi_L \ln n \qquad (28)$$

and can be calculated easily.

We now consider the random and tree percolation separately.

5.1. Random percolation

We must set $n = 1$, or $\nu = 0$. This simplifies the polynomial Q_1, and we find that x, x_1 and x_2 are independent of l, and are given by equation (14). The combination

$$\xi = u^2/K(K+2) \qquad (29)$$

is also independent of l. It is easy to see that the free energy per site ω from equation (27) is $2\ln u$, which is expected[2]; see equation (3). The bond density is

$$\phi_{B,RP} = 2K/u \qquad (30)$$

and the probability p that a bond is occupied, see equation (2c), is given by $p_{RP} \equiv \phi_{B,RP}/2 = K/u$ in random percolation [11,12]. In terms of the bond density, we have $K = \phi_B/(2 - \phi_B)$, and we can use equation (28) to obtain the entropy per site

$$S(\phi_B) = -2[p \ln p + (1-p)\ln(1-p)]$$

where p is the bond occupation probability. The entropy *per lattice* bond is just the standard combinatorial entropy of choosing B bonds out of N_B lattice bonds. The free energy per bond is $\ln[1/(1-p)]$ in random percolation.

The behaviour in the limit $K \to \infty$ is easy to explore. In this limit, all bonds on the lattice must be occupied and maximum possible loops due to polymer bonds must be formed. We call this limiting cluster the maximal cluster. For $K \to \infty$, x and x_1 vanish so that $\xi \to 1$ and $x_2 \to 1$. Indeed, we see from equations (21), and (22) that only $\phi_4 = 1$ is non-zero. Thus, the monomer density is one, and there are no solvent molecules. Furthermore, the bond density

---

[2] The free energy $\omega = \ln(1 + K)$ in [11] is per bond and not per site.

$\phi_B = 2$, which is its maximum possible value consistent with $q = 4$. From equation (26), we also find that $\phi_L = 2/l$.

We now apply equation (2b) to the maximal cluster in the limit $K \to \infty$ to calculate the *total* density $\phi_{L,\text{tot}}$ of loops. Since all sites are covered, we have $\phi_M = 1$ and $\phi_P = 0$ (only one maximal cluster). Thus, $\phi_{L,\text{tot}} = 1$, which is greater than $\phi_L = 2/l$ calculated above. The latter represents the density of all *finite* loops in the maximal cluster, which is an *infinite* cluster. The difference is due to *infinite loops* that have appeared after percolation, as discussed in [11]. The density of infinite loops in the maximal cluster has its *maximum* value and is given by

$$\phi_{\infty L,\text{max}}^{(\infty)} = (l - 2)/l \tag{31}$$

and increases with l. We will see below that it is also the loop density in the maximal cluster in the tree percolation. The infinity symbol in the subscript is a reminder that we are considering infinitely large loops, while in the superscript it is a reminder that we are considering an infinite cluster. On the other hand, the finite loop density decreases. (It should be recalled from [11] that equation (2b) is applicable to finite and infinite clusters separately.) For a triangular Husimi cactus, we have $\phi_{\infty L,\text{max}}^{(\infty)} = 1/3$ for infinitely large loops, and $\phi_{fL,\text{max}}^{(\infty)} = 2/3$ for finite-sized loops. This result was quoted in [11] without any proof. For a square cactus, they are equal to $\frac{1}{2}$. For comparison, we have $\phi_{\infty L,\text{max}}^{(\infty)} = 1$ and $\phi_{fL,\text{max}}^{(\infty)} = 0$ on a $q = 4$ Bethe lattice.

5.2. Tree percolation

In this case, we must set $n = 0$, or $\nu = -1$. This destroys the simple nature of Q_1, and the problem becomes somewhat complicated. At the FP, we have

$$x = u^{l-2}/K[(K + 2)u^{l-2} - K^{l-1}] \qquad x_1 = 2u^{l-2}/[(K + 2)u^{l-2} - K^{l-1}]$$
$$x_2 = K(u^{l-2} - K^{l-2})/[(K + 2)u^{l-2} - K^{l-1}] \tag{32}$$

and the solution is *no longer* independent of the polygon size l. From equation (24), we find

$$\phi_{B,TP} = 2K \left(\frac{u^{l-1} - K^{l-1}}{u^l - K^l} \right) \tag{33}$$

with the probability of a bond being occupied given by $p_{TP} \equiv \phi_{B,TP}/2$. We see immediately that the bond density in the tree percolation is *smaller* than its value in random percolation at the same K.

We again consider the $K \to \infty$ limit. We find that x vanishes, but

$$x_1 \to 2/l \qquad x_2 \to (l - 2)/l. \tag{34}$$

Therefore, the solvent and the end-point densities vanish, as expected. However, the other functionalities do not vanish:

$$\phi_2 \to 4/l^2 \qquad \phi_3 \to 4(l - 2)/l^2 \qquad \phi_4 \to (l - 2)^2/l^2. \tag{35a}$$

The bond density, therefore, from the first equation in (2b) is

$$\phi_{B,TP} \to 2(l - 1)/l < 2 \tag{35b}$$

and is less than its value in random percolation, which is *always* two. The density in equation (35b) is the maximum possible value when *no* loops are formed inside the polygons, as can easily be checked. The finite size loop density $\phi_{fL}^{(\infty)}$ is obviously zero as can be seen from equation (26); here, the subscript f stands for finite-size loops. To calculate the infinite-size loop density in the maximal infinite cluster, we again turn to the last equation in (2b) and use the fact that the monomer density is one. We find that it is exactly the density in equation (31).

Thus, the infinite-size loop density in the maximal cluster does not depend on the value of the loop activity. This is not surprising as the number of loops reaches its asymptotic value, and n plays no role in this limit.

6. Percolation

We now proceed with the calculation of percolation properties. We restrict ourselves to the FP solution, and introduce the probability R that the cluster of connected polymer bonds, which reaches the mth level from higher levels, is a *finite* cluster above this level. Since a cluster has reached this level, there must be at least one bond incident on this level from above. Therefore, the base site at the mth level is in the b-state. We now consider all possible states t of the upper sites and look for the condition that this cluster remains finite above any of these upper sites. To find the contribution $Z_m^{(f)}(b)$ of all clusters that *reach* the mth level from *above*, but are finite above this level, it will be convenient to consider all possible ways upper sites are connected to the base site. For this, we assume that the base site is connected to upper sites by a linear chain or a ring of d occupied bonds; thus, d may be equal to l. Every upper site on this chain or ring cannot have a solvent molecule. The remaining sites left uncovered by the chain in the upper polygon may be connected together by polymer bonds. However, these bonds must be *disjoint* from the above chain.

We first consider the triangular cactus and the upper sites in the mth triangle. If an upper site is in the s- or 0-state, the cluster is finite above that site with probability one. If a site is in the b-state, it is finite above that site with a probability R. However, we need to include the probability R for only those b-state upper sites that are connected to the mth level base site. For upper sites in the b-state that are disconnected from the base site, the cluster is finite above the base site with probability one.

Single incident bond. Some of the possible states of the triangles are shown in figures 4(a), (b). Let us take $d = 1$. First consider figure 4(a). One of the upper sites is in the s-state which, therefore, is not connected to the mth level site. The other site can be in either the 0- or in the b-state, and is connected with the mth site. The contribution from $Z_{m+1}(b)$ must be reduced by R. In the following, we will find it convenient to introduce the following combination:

$$\overline{U}_{m+1} = Z_{m+1}(0) + R Z_{m+1}(b) \tag{36}$$

which reduces to U_{m+1} for $R = 1$. The above contribution from finite clusters ($d = 1$ chain) is $2 K Z_{m+1}(s) \overline{U}_{m+1}$. We can replace the s-state by a b-state, such that this site remains disconnected from the base site. This gives the additional contribution $2 K Z_{m+1}(b) \overline{U}_{m+1}$. It is *not* possible to replace the s state by a 0 state for $d = 1$. Adding the two contributions, we find that the total contribution from finite clusters with the $d = 1$ chain is $2 K V_{m+1} \overline{U}_{m+1}$. For the $d = 2$ chain with two bonds connecting the upper sites with the mth site, we need to consider figure 4(b), in which we can replace each 0-state by a b-state also. The total finite cluster contribution from these states is similarly $2 K^2 \overline{U}_{m+1}^2$.

Double incident bonds. We need to consider figures 4(c), (d). In figure 4(c), we can again replace each 0 state by a b state. Thus, the contribution is $K^2 \overline{U}_{m+1}^2$, as above for figure 4(b). For the closed loop formed by three bonds ($d = l$), figure 4(d), the contribution is $K^3 n \overline{U}_{m+1}^2$.

The contributions to $Z_m^{(f)}(b)$ from the closed loop and a chain with $d = l - 1$ contain the same power ($l' \equiv l - 1$) of $\overline{U}_{m+1}$; the pre-factors $K^l n$ and $l K^{l'}$, respectively, are different. We also conclude that each upper level site connected to the mth site contributes a factor of $\overline{U}_{m+1}$,

and each upper level site disconnected from the mth site contributes a factor of V_{m+1} provided there is no bond in the polygon connected to this site. For the triangular cactus, we get

$$Z_m^{(f)}(b) = 2K V_{m+1}\overline{U}_{m+1} + 3K^2\overline{U}_{m+1}^2 + K^3 n\overline{U}_{m+1}^2. \tag{37}$$

By definition, we have

$$R = Z_m^{(f)}(b)/Z_m(b). \tag{38a}$$

However, it is convenient to introduce another more useful quantity $\overline{R}$ defined as below:

$$\overline{R} = [Z_m^{(f)}(0) + Z_m^{(f)}(b)]/[Z_m(0) + Z_m(b)] \tag{38b}$$

where we must observe that $Z_m^{(f)}(0)$ and $Z_m(0)$ are the same, as a site in 0-state is always part of a finite cluster above it. The same is true of an s-state site. The above combination is more useful since the denominator U_m in equation (38b) will be a common factor as we will see below.

Dividing the numerator and the denominator on the right-hand side in equation (38b) by $Z_m(b)$, we find that

$$\overline{R} = (x + R)/(x + 1) = \zeta/\xi \tag{39}$$

where we have introduced a new combination $\zeta(R) \equiv x + R$, so that $\xi \equiv \zeta(1)$, used in equation (11b), and:

$$\overline{U}_{m+1}/Z_{m+1}(b) = \zeta. \tag{40}$$

Using equations (37) in (38b), we immediately find the following FP equation determining $\overline{R}$:

$$\overline{R} = \rho(\overline{R}) \equiv x/\xi + \xi[2K\overline{R} + (3K^2 + nK^3)\overline{R}^2]/Q_1 \tag{41}$$

where Q_1 is given in equation (12) with $l = 3$. We note that at $R = 1$, $\overline{R} = 1$, which is always a solution of equation (41).

The knowledge of R or $\overline{R}$ is important to calculate separate contributions to any thermodynamic density from finite and infinite clusters. According to the fundamental principles laid down in [11], the k-functional densities in the finite and infinite clusters are, respectively, given by

$$\phi_k^{(f)} = \overline{\phi}_k \qquad \phi_k^{(\infty)} = \phi_k - \overline{\phi}_k \tag{42a}$$

where we have introduced new quantities $\overline{\phi}_k$ as in equations (20)–(22), except that we must replace x_k by $\overline{x}_k$ on the right-hand side, where

$$\overline{x} = x \qquad \overline{x}_1 = x_1 R_1 \qquad \overline{x}_2 = x_2 R_2. \tag{42b}$$

The quantity

$$R_k = Z_m^{(f)}(k)/Z_m(k) \tag{42c}$$

denotes the probability that the k-state ($k = 1, 2$) at the mth site is part of a finite cluster at higher levels. A similar quantity for $\sigma = 0$ or s is evidently one, as either one of them is unconnected to a cluster above the base site. It is easily seen that

$$R_1 = \overline{R}(1 + K\overline{R})/u \qquad R_2 = \overline{R}^2 \tag{43a}$$

and satisfy the following relation:

$$\overline{R} = (x + R_1 x_1 + R_2 x_2)/\xi \tag{43b}$$

which is equivalent to equation (41) as can be easily checked. We note that for $R = 1$, all other R are also one. Thus, all R simultaneously become different from one, and any one of them can be used to signal percolation in the system.

We can apply equation (2b) to finite and infinite clusters separately. Thus, we can calculate various densities for the finite and the infinite clusters. Let us consider the contributions from finite clusters to the monomer and bond densities. A simple algebra yields

$$\phi_M^{(f)} = \overline{R}^2 - (x/\xi)^2 \qquad \phi_B^{(f)} = \overline{R}(\overline{x}_1 + 2\overline{x}_2)/\xi. \qquad (44a)$$

The form of these expressions should be compared with those in equation (23). The contributions from the infinite cluster are obtained by subtracting these quantities from their total values ϕ_M, ϕ_B respectively. A simple algebra yields

$$\phi_M^{(\infty)} = 1 - \overline{R}^2 \qquad \phi_B^{(\infty)} = [x_1(1 - R_1) + 2x_2(1 - R_2)]/\xi. \qquad (44b)$$

For the triangular cactus, we must use equation (43a) in the second of equations (44b). It is evident that $\phi_k^{(\infty)}$, $\phi_M^{(\infty)}$ and $\phi_B^{(\infty)}$ vanish, as expected, before percolation since $\overline{R} = 1$. However, as soon as percolation occurs, all these quantities become *nonzero*.

The percolation probability $\mathsf{P}_n(p)$ is related to the bond density of the infinite cluster as follows:

$$\mathsf{P}_n(p) \equiv \phi_B^{(\infty)}/\phi_B = [x_1(1 - R_1) + 2x_2(1 - R_2)]/(x_1 + 2x_2). \qquad (44c)$$

We turn back to equation (41). Graphically, we look for the crossing of $\rho(\overline{R})$ and $\overline{R}$ to find the solution. We immediately notice that $\overline{R} = 1$ is *always* a solution. This solution corresponds to all clusters being finite. For an infinite cluster to appear, a new solution $\overline{R} < 1$ must appear. (A solution $\overline{R} > 1$ is unphysical as a probability and must always be rejected.) We refer the reader to [12] for a detailed graphical analysis.

The slope $\rho'(\overline{R})$ at $\overline{R} = 1$ is

$$\rho'(1) = \xi(2K + 6K^2 + 2nK^3)/Q_1 \qquad (45)$$

and is usually less than one for small K. Thus, for small K, there must be an unphysical solution $\overline{R} > 1$ which must be rejected. There is only one solution $\overline{R} = 1$ for small K, and all clusters are *finite*. For large enough K, the slope at $\overline{R} = 1$ becomes larger than one. When this happens, there will be a crossing between $\rho(\overline{R})$ and $\overline{R}$ at a smaller value of $\overline{R}$. This new solution corresponds to the presence of an infinite cluster.

It is clear that the percolation threshold occurs when $\rho'(1)$ is exactly one. This condition is the same as on a Bethe lattice [12]. Thus, the critical value K_c for percolation to occur is given by

$$nK^3 + 3K^2 - K - 1 = 0. \qquad (46a)$$

For random percolation, it is convenient to express this equation in terms of the probability $p = K/u(1 - p = 1/u)$ that a lattice bond is occupied:

$$2p^3 - 2p^2 - 2p + 1 = 0 \qquad (46b)$$

which is the known result for random percolation on a triangular cactus [6], for which the critical values are given by

$$p_{c,RP}^{(\text{triangle})} \cong 0.403\,03 \qquad K_{c,RP}^{(\text{triangle})} \cong 0.675\,13. \qquad (47a)$$

For the tree percolation, we go back to equation (46a). We find that the critical activity and the critical probability are given by

$$K_{c,TP}^{(\text{triangle})} = (\sqrt{13} + 1)/6 \cong 0.767\,59 \qquad p_{c,TP}^{(\text{triangle})} \cong 0.383\,80 \qquad (47b)$$
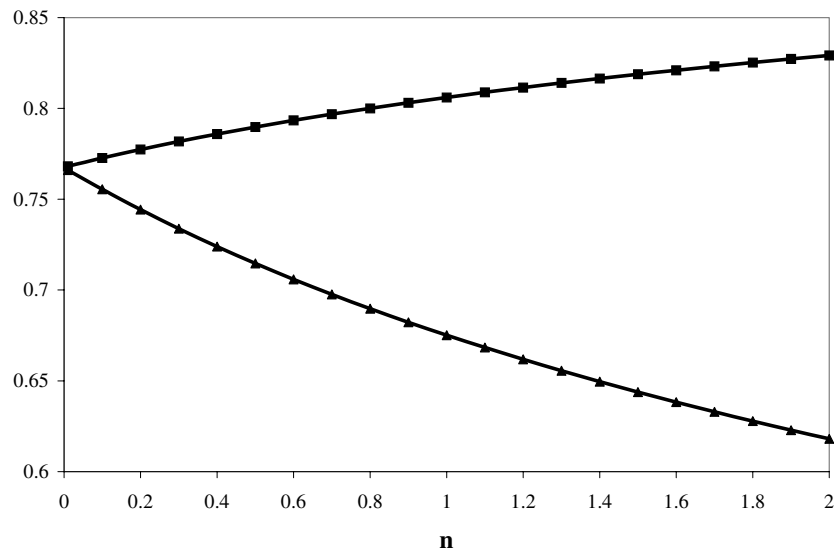
Figure 5. The critical bond activity $K_c(n)$ (▲) and the critical bond density $\phi_{B,c}(n)$ (■) as a function of n.

where we have used equation (33) to calculate the bond density ϕ_B, the probability p being half of it. From equation (46a), we immediately conclude that $K_c(n)$ is a monotonically decreasing function of n. We show $K_c(n)$ (▲) and $\phi_{B,c}(n)$ (■) in figure 5 as a function of n. We thus note that $p_c(n) = \phi_{B,c}(n)/2$ depends on the loop activity n and *increases* monotonically. (On a Bethe lattice, the critical probability for $q = 4$ is $p_c^{(\text{Bethe})} = 1/3$, independent of n.)

Percolation occurs *earlier* in the tree percolation than in random percolation, as one would expect, as there is *no* wastage of bonds in forming loops without increasing the size of the cluster. What is interesting to observe is that the tree percolation occurs earlier on the triangular cactus than on a four-coordinated Bethe lattice. Immediately after percolation, the infinite cluster in the tree percolation will have some infinitely large loops, and their density will be non-zero, just as was the case on a Bethe lattice as shown in [11]. Thus, for p between 0.282 87 and 0.403 03, there are infinitely large loops in tree percolation, but not in random percolation. This establishes that the density of infinitely large loops depends on the loop activity n.

7. Percolation for general l

We now turn to percolation study for arbitrary loop size. For this, it will be useful to consider a square cactus and extend the above analysis. We first remark that the case $d = l'$ is different from $d < l'$ in that the latter allows the possibility of connecting the remaining sites with bonds and still have these sites remain disconnected from the base site. Thus, we will consider the two cases separately. This distinction can only be made for $l \geqslant 4$. The case in which all sites are connected to the base site requires either a closed loop of l bonds, or a maximal chain of length $d = l'$. The former case requires an extra factor n of the loop activity. The contribution from the closed loop to $Z_m^{(f)}(b)$ is simply $nK^l\overline{U}_{m+1}^{l'}$. The maximal chain covering all upper sites contributes $lK^{l'}\overline{U}_{m+1}^{l}$; the prefactor l accounts for the lattice bond not occupied.

We now consider the remaining possibility $d < l'$. There are $j = l - d - 1$ upper sites

that are *disconnected* from the base site, but may be connected among themselves by polymer bonds. We focus on these sites. Each such site contributes V_{m+1} if it is not connected by a bond, or U_{m+1} if it is connected by a bond. For $j = 1$, the contribution is V_{m+1}, as no bond can be connected to it and still keep it disconnected from the base site. For $j = 2$, the contribution from the two sites is $V_{m+1}^2 + K U_{m+1}^2 = u U_{m+1}^2$, the factor of K coming from the presence of a bond. It is easy to see that the contribution from the j sites is $u^{j-1} U_{m+1}^j$. The contribution from the chain of d bonds is, as before, $(d+1) K^d \overline{U}_{m+1}^d$, the first factor taking into account all possible ways this chain passes through the base site. Thus, we finally have

$$Z_m^{(f)}(b) = (nK^l + lK^{l'})\overline{U}_{m+1}^{l'} + \sum_{d=1}^{l-2}(d+1)K^d\overline{U}_{m+1}^d u^{j-1} U_{m+1}^j. \tag{48a}$$

We can also calculate the two parts $Z_m^{(f)}(1)$ and $Z_m^{(f)}(2)$ of $Z_m^{(f)}(b)$. We merely quote the result:

$$Z_m^{(f)}(1) = 2\left[K^{l'}\overline{U}_{m+1}^{l'} + \sum_{d=1}^{l-2} K^d\overline{U}_{m+1}^d u^{j-1} U_{m+1}^j \right]$$

$$Z_m^{(f)}(2) = [nK^l + (l-2)K^{l'}]\overline{U}_{m+1}^{l'} + \sum_{d=1}^{l-2}(d-1)K^d\overline{U}_{m+1}^d u^{j-1} U_{m+1}^j. \tag{48b}$$

Using equations (11b) and (40), we can show that

$$Z_m^{(f)}(b)/Z_m(b) = \left(\frac{\xi^{l-1}}{Q_1} \right)\left((nK^l + lK^{l-1})\overline{R}^{l-1} + u^{l-2}\sum_{d=1}^{l-2}(d+1)(K\overline{R}/u)^d \right). \tag{49}$$

It is easy to check that the right-hand side of equation (49) reduces to one for $\overline{R} = 1$. The sum in the equation is the derivative $f'(y)$ of the following sum:

$$f(y) = \sum_{d=2}^{l-1} y^d = y^2(1 - y^{l-2})/(1 - y)$$

where we have used the fact that $y \equiv K\overline{R}/u < 1$ for all finite K, and approaches $\overline{R}$ as $K \to \infty$. The derivative is trivially calculated:

$$f'(y) = \sum_{d=1}^{l-2}(d+1)y^d = (2y - y^2 - ly^{l'} + l'y^l)/(1 - y)^2.$$

We finally have the equation determining $\overline{R}$ from equation (38b):

$$\overline{R} = \rho(\overline{R}) \equiv x/\xi + \left(\frac{u^l}{u^l + vK^l} \right)[(nK + l)y^{l-1}/u + (1/u^2)f'(y)]. \tag{50a}$$

We also express R_1 and R_2 in terms of $\overline{R}$ by using equation (48b). We have

$$x_1 R_1 = 2\left(\frac{\xi^{l-1}}{Q_1} \right)\left(K^{l-1}\overline{R}^{l-1} + u^{l-2}\sum_{d=1}^{l-2}(K\overline{R}/u)^d \right)$$

$$x_2 R_2 = \left(\frac{\xi^{l-1}}{Q_1} \right)\left([nK^l + (l-2)K^{l-1}]\overline{R}^{l-1} + u^{l-2}\sum_{d=1}^{l-2}(d-1)(K\overline{R}/u)^d \right). \tag{50b}$$

It is easily seen that equation (50b) reduces to (43a) for $l = 3$. The percolation condition is given by $\rho'(\overline{R} = 1) = 1$, which is equivalent to

$$[1 + (l-2)n - ll'](K/u)^l + ll'(K/u)^{l-1} + (K/u^3)f''(K/u) = 1 \tag{51}$$

where

$$f''(p) = \sum_{k=1}^{l-2}(k+1)kp^{k-1}.$$ (52)

The relation between $\overline{R}$, R_1 and R_2 is given in equation (43*b*), which is valid for all l. Similarly, equations (42*a*), (44*a*) and (44*b*) remain valid for all l. This solves the problem for general l. For $l = 3$, equation (51) reduces to (46*a*).

To calculate the loop density of (finite) loops of size l in finite clusters, we follow the derivation of the loop density in equation (26*a*), except that each U_m in the numerator must be replaced by $\overline{U}_m$. Consequently, we find

$$\phi_L^{(f)} = 2nK^l\zeta^l/lQ_1\xi^2 = \frac{2n(K\overline{R})^l}{l(u^l + K^l v)}.$$ (53*a*)

Let us apply the last of equations (2*b*) to finite clusters to calculate the number density of finite clusters, which is given by

$$\phi_P^{(f)} = \phi_M^{(f)} - \phi_B^{(f)} + \phi_L^{(f)}.$$ (53*b*)

The difference $\phi_L^{(\infty)} = \phi_L - \phi_L^{(f)}$ is the density of finite-size loops ($\phi_{fL}^{(\infty)}$) in the infinite cluster, since ϕ_L contains only finite loops. To compute the density of infinite, i.e. macroscopic, loops ($\phi_{\infty L}^{(\infty)}$), we proceed to apply the last of equations (2*b*) to the infinite cluster whose number density is zero. Thus,

$$\phi_{\infty L}^{(\infty)} = \phi_B^{(\infty)} - \phi_M^{(\infty)} - \phi_{fL}^{(\infty)}$$ (53*c*)

and can be obtained by the use of equation (44*b*). It is obvious that the *macroscopic* loop density is non-zero as soon as the infinite cluster appears, and is zero before percolation. Thus, the macroscopic loops appear immediately after percolation, as was also the case on a Bethe lattice. Indeed, the contribution from an infinite cluster to any density becomes non-zero as soon as percolation occurs.

We now consider the limiting behaviour as $l \to \infty$. We show that the solution reproduces the result for a $q = 4$ coordination number Bethe lattice. We recall that $y < 1$ for finite K. Hence, y^l or $y^{l'}$ vanishes and $f(y) \to y^2/(l-y)$, and there is no dependence on n, which is a property of the solution on a Bethe lattice. Thus, we can replace K/u by p. We find in this limit that $f'(y) = y(2-y)/(1-y)^2$, $f''(p) = 2u^3 = 2/(1-p)^3$ and $x/\xi \to 1/u_2$. The first term inside the square bracket in equation (50*a*) also vanishes in this limit. Hence, $\overline{R}$ is given by

$$\overline{R} = [(1-p)/(1-y)]^2.$$ (54)

Similarly, the first two terms in equation (51) also vanish and the condition for critical percolation is given by the last term, which is

$$2p = 1 - p.$$ (55)

Thus, the critical percolation probability $p_c = 1/3$, which is a known result for a four-coordinated Bethe lattice. We also note from equation (31) that the loop density of the infinitely large loops in the infinite cluster becomes one. This is also the loop density of the infinitely large loops on a $q = 4$ coordinated Bethe lattice, as discussed at the end of section 5.1. The density of finite size loops is certainly zero.

It should be obvious that the coefficient 2 on the left-hand side of equation (55) is due to the two bonds incident on the base site of a polygon. If we consider a cactus in which δ upper polygons are incident, then this factor will be replaced by 2δ in equation (55). Then the critical

percolation probability will be given by the well known result for a $2(\delta+1)$-coordinated Bethe lattice:

$$p_c^{(\text{Bethe})} = 1/(2\delta + 1). \tag{56}$$

8. Conclusions

We have considered correlated percolation, which includes random and tree percolation, on a Husimi cactus with arbitrary l in order to shed light on the effects of loops on percolation. The loop activity n is a parameter in the model. The solution is obtained recursively and is *exact* on the cactus. We conclude that infinitely large loops appear immediately after percolation, similar to the situation encountered on a Bethe lattice [11, 12]. Following the arguments presented in [11, 12], we easily conclude that these *macroscopic* loops are formed by the polygons at the surface of the cactus in the limit when it becomes infinitely large. The density of such macroscopic loops depends on the value of the loop activity n, and is non-zero even for tree percolation. We have calculated the contributions to all densities from *finite* and *infinite* clusters in the recursive scheme. The percolation threshold p_c and the percolation probability $P_n(p)$ are found to be functions of l and n, and the general results are given in equations (51), and (44c), respectively. We have shown that the critical bond probability for tree percolation is *smaller* than its value for random percolation. Indeed, it is a monotonically increasing function of the loop activity n, as shown in figure 5. This is consistent with the common belief that the presence of loops only helps to reduce the size of a cluster containing the same number of bonds. The critical threshold and the density of macroscopic and finite loops, if they are present, for the tree and the random percolation depend on the size l of the polygon.

An interesting property of the random percolation is the observed *independence* of the ratios x and x_1 of the size l of the polygons. This is a reflection of the inherent randomness in the placement of bonds on any lattice. Thus, l plays no role in random percolation as far as the local densities ϕ_k, ϕ_M and ϕ_B are concerned. Even the density of loops $\phi_L = (2/l)p^l$, see equation (26), depends on l in a trivial fashion. The factor of l in the denominator is due to the definition of ϕ_L being per site, and will disappear if we define it per polygon. The factor p^l is trivial and accounts for the presence of l bonds in each loop. However, percolation properties like the percolation threshold, percolation probability, the density of macroscopic loops, etc depend on l.

Comparing the random percolation on a $q = 4$ Bethe lattice with that on a Husimi cactus, also having $q = 4$, we observe an interesting phenomenon. The critical threshold on the Bethe lattice is $p_c^{(\text{Bethe})} = 1/3$, and the density of macroscopic loops in the maximal cluster [12] is 1 on the $q = 4$ Bethe lattice, where there is no distinction between the tree percolation and random percolation. On a cactus, $p_c \cong 0.403\,03$ for $l = 3$, and decreases as l increases, reaching $p_c^{(\text{Bethe})} = 1/3$ as l diverges. The critical probability for tree percolation on a triangular cactus is smaller than that for random percolation on the cactus, but larger than that on the Bethe lattice, and approaches the latter as l diverges. The critical probability on the Bethe lattice is a lower bound for general (arbitrary n) percolation, and the two approach each other as l diverges. The macroscopic loop density in the maximal cluster on the cactus is always less than one, see equation (31), and approaches one as l diverges.

The limit $l \to \infty$ for the cactus is very interesting. In this limit, the polygons become infinitely large, and the cactus looks identical to a $q = 4$ Bethe lattice, and the *connectivity* of the polygon becomes inconsequential. At a microscopic level, there does not appear any difference between the Bethe lattice and the cactus. From equation (26), we note that in the

$l \to \infty$ limit, the loop density vanishes, since $K/u < 1$ for all finite K. This is consistent with what is expected on a Bethe lattice. The density of macroscopic loops is also equal, as noted in the previous paragraph. The percolation thresholds are also the same.

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